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ELASTO-PLASTICITY OF THE CLUB SANDWICH

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Abstract—The kinematics and dynamics of thin shells are well established. The constitutive equations of Hookean shells are linear in 6 strains and 6 stresses, but the equations of elastoplastic shells are incremental and require additional internal variables, notably stresses. In typical computations, the shell is divided into N layers: With the usual hypotheses, 6 strains and $3N$ stresses are required. The storage of $3N$ stresses poses practical limitations. The ideal sandwich of 2 layers is useful for limit analyses, but inadequate for general purposes. The club sandwich of 4 layers offers a practical alternative: Relative strengths, stiffnesses and positions of the layers are selected to fit the conditions of yielding under actions of membrane and bending stresses. Here, the mechanics of the club sandwich are presented and the behavior is compared with that of the multi-layered shell.

NOTATIONS

y	distance from middle surface
d, c	distance from middle surface to intermediate, outer layer
z	$z \equiv y/c$
α	$\alpha \equiv c/d$
β	$\beta \equiv t_1/t_0$
t_1, t_0	thickness of intermediate, outer layer
E	Young's modulus of elasticity
Y	yield stress in simple tension
$\tau^{\alpha\beta}$	physical component of stress $\alpha = 1, 2$; Greek indices signify surface coordinates
$\gamma_{\alpha\beta}$	physical component of strain
k_1, k_0	stiffness (E_1/E, E_0/E) of intermediate, outer layer
$M_N{}^{\alpha\beta}$	physical component of resultant (force $M_0{}^{\alpha\beta}$, couple $M_1{}^{\alpha\beta}$)
$\sigma^{\alpha\beta}$	$\sigma^{\alpha\beta} \equiv \tau^{\alpha\beta}/Y$
$n_N{}^{\alpha\beta}$	$n_N{}^{\alpha\beta} = \dfrac{M_N{}^{\alpha\beta}}{2(1+\beta)t_0 Y c^N}$
$\epsilon_{\alpha\beta}$	$\epsilon_{\alpha\beta}$ $\epsilon_{\alpha\beta} \equiv \gamma_{\alpha\beta}E/Y$
i	index $i = 0, 1, 2$ or 3 denotes layer as labeled in Fig. 1

The dimensionless stress $n_1{}^{\alpha\beta}$ and strain $\epsilon^1_{\alpha\beta}$ differ from their counterparts, $m_1{}^{\alpha\beta}$ and $\kappa_{\alpha\beta}$ of Ref. [6].

$$n_1{}^{\alpha\beta} = \frac{1}{\sqrt{3}} m_1{}^{\alpha\beta}, \qquad \epsilon^1_{\alpha\beta} = \sqrt{3}\kappa_{\alpha\beta}.$$

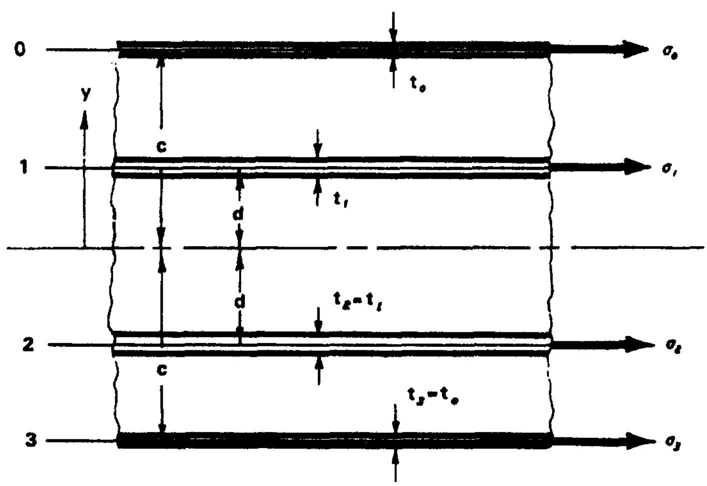

Fig. 1.

INTRODUCTION

In the first approximation of a shell, the deformation is described by two fundamental tensors of a reference surface, the metric and curvature tensors. The changes in these tensors determine two symmetric tensors of strain, $\epsilon^0_{\alpha\beta}$ and $\epsilon^1_{\alpha\beta}$. Then the power $\dot{w}$ of stresses establishes two tensors of stress, $n_0{}^{\alpha\beta}$ and $n_1{}^{\alpha\beta}$:

$$\dot{w} = n_0{}^{\alpha\beta}\dot{\epsilon}^0_{\alpha\beta} + n_1{}^{\alpha\beta}\dot{\epsilon}^1_{\alpha\beta}.$$

The strain tensors are precisely determined by the displacement of the reference surface[1, 2] and then, the equations of equilibrium are determined by the principle of virtual work. Hence, the kinematical and dynamical equations of the first approximation may hold independently of the constitutive equation. Accordingly, we may proceed with caution to seek approximations of the latter, specifically the stress–strain ($\mathbf{n}, \boldsymbol{\epsilon}$) equations of the elastic–plastic shell.

Our problem is nonconservative, so that the only algebraic equations relating stresses ($\mathbf{n}$) and strains ($\boldsymbol{\epsilon}$) are incremental. Indeed, the arguments of classical plasticity are applicable to our two-dimensional problem; if a yield condition $f(\mathbf{n})$ is known, then an increment of plastic strain is governed by the normality condition, $\dot{\epsilon}^{NP}_{\alpha\beta} = \dot{\lambda}\partial f/\partial n^N_{\alpha\beta}$. Such a direct procedure has been used most effectively by Bieniek[3].

A derivation of the requisite constitutive equations from three dimensions is complicated by the progression of plastic strain through the thickness. Nonetheless, deformational theories have been developed[4, 5]. An incremental theory was given by the author in a recent paper[6] which also provides a brief historical account of other developments.

The present paper describes the behavior of a simple conceptual model: The shell is composed of 4 discrete layers, each exhibiting ideal isotropic elasticity and plasticity according to the yield criterion of von Mises. The model exhibits a discrete progression of yielding from layer-to-layer; the consequent stress–strain curves are piecewise linear. The theory and computations indicate that our "club sandwich" is a useful approximation, though the errors may be unacceptable in certain problems which are sensitive to imperfections. Additionally, the study offers some guidelines to alternative approximations.

BASIC APPROXIMATIONS

Certain assumptions are drawn from our experience with other first approximations: Transverse stress components are neglected; the strain components vary linearly through the thickness of the sandwich:

$$\epsilon_{\alpha\beta} = \epsilon^0_{\alpha\beta} + z\epsilon^1_{\alpha\beta}. \tag{1}$$

The plasticity is governed by the Mises condition and the associated flow rule, respectively:

$$f = \tfrac{3}{2}(\sigma^{\alpha\beta}\sigma_{\alpha\beta} - \tfrac{1}{3}\sigma_\alpha{}^\alpha\sigma_\beta{}^\beta) = 1 \tag{2}$$

$$\dot{\sigma}^{\alpha\beta} = (C^{\alpha\beta\gamma\eta} - B^{\alpha\beta\gamma\eta})\dot{\epsilon}_{\gamma\eta}. \tag{3}$$

The parenthetical term of the last equation is the tangential modulus which follows from the normality and loading conditions:

$$C^{\alpha\beta\gamma\eta} = \frac{1}{1+\nu}\left(\delta^{\alpha\gamma}\delta^{\beta\eta} + \frac{\nu}{1-\nu}\delta^{\alpha\beta}\delta^{\gamma\eta}\right) \tag{4a}$$

$$B^{\alpha\beta\gamma\eta} = \frac{C^{\alpha\beta\mu\phi}C^{\gamma\eta\kappa\zeta}S_{\mu\phi}S_{\kappa\zeta}}{D} \tag{4b}$$

$$D = C^{\alpha\beta\gamma\eta}S_{\alpha\beta}S_{\gamma\eta} \tag{4c}$$

$$S^{\alpha\beta} = \sigma^{\alpha\beta} - \tfrac{1}{3}\sigma_\eta{}^\eta\delta^{\alpha\beta}. \tag{4d}$$

Here, all quantities are nondimensional as defined in "Notations". Upper and lower indices are employed only to facilitate the transition to arbitrary coordinates; here, all are treated as

Cartesian, that is, curvatures of the reference surface are neglected. The coefficient $C^{\alpha\beta\gamma\eta}$ is the elastic modulus and $S^{\alpha\beta}$ is the deviator of stress.

CLUB SANDWICH

The model of 4 discrete layers, symmetric about the middle surface, is depicted in Fig. 1. Four parameters characterize the layers: The relative thickness and position of intermediate layers are defined by 2 parameters,

$$\alpha = \frac{c}{d}, \qquad \beta = \frac{t_1}{t_0}. \tag{5a, b}$$

The stiffnesses of the inner and outer layers are characterized by 2 parameters

$$k_0 = \frac{E_0}{E}, \qquad k_1 = \frac{E_1}{E}. \tag{5c, d}$$

A miniscule Latin subscript $i = 0, 1, 2$ or 3 identifies the layer as labeled in Fig. 1. In our model the properties of the outer layers (0, 3) are the same; the properties of the inner layers (1, 2) are the same.

Components of the nondimensional resultants follow:

$$n_N^{\alpha\beta} \equiv \frac{1}{2(1+\beta)} \sum z_i^N \left(\frac{t_i}{t_0}\right) \sigma_i^{\alpha\beta}. \tag{6}$$

If four resultants $(n_N^{\alpha\beta})$ are defined $(N = 0, 1, 2, 3)$, then the four stresses $(\sigma_i^{\alpha\beta})$ $(i = 0, 1, 2, 3)$ can be expressed in terms of these resultants and also the yield condition of each layer.

Now, the parameters must be chosen such that the model exhibits the essential features of the homogeneous plate (or shell). Specifically, we require the correct initial moduli under the action of force and moment, respectively; in the nondimensional variables,

$$\frac{\mathrm{d}n_0{}^{11}}{\mathrm{d}\epsilon_{11}^0} = 1, \qquad \frac{\mathrm{d}n_1{}^{11}}{\mathrm{d}\epsilon_{11}^1} = \frac{1}{3}. \tag{7a, b}$$

Also, we require the correct limit moment in bending; here, $n_1{}^{11} = 1/2$ when $\sigma_0{}^{11} = \sigma_3{}^{11} = -\sigma_1{}^{11} = -\sigma_2{}^{11}$:

$$\frac{1}{2} = \left(\frac{1+\beta/\alpha}{1+\beta}\right). \tag{7c}$$

Finally, the value of the moment at initial yielding in bending gives the value $n_1{}^{11} = 1/3$. If we adhere to this condition, then the 4 parameters $(\alpha, \beta, k_0, k_1)$ are fixed and the constitutive equations are fully determined. Instead, we admit a small disparity (ϵ) and set the initial yielding at the moment,

$$n_1{}^{11} = \frac{1+\epsilon}{3} = \frac{1}{1+\beta}\left[1 + k_1 \frac{\beta}{\alpha^2}(1+\epsilon)\right]. \tag{7d}$$

The four eqns (7a)–(d) can be solved to express the four parameters $(\alpha, \beta, k_0, k_1)$ in terms of the unspecified parameter ϵ. The latter is selected to provide the best fit for the anticipated conditions of loading. If membrane forces dominate, then $0 \le \epsilon \ll \frac{1}{2}$. If bending dominates, then a positive $\epsilon (< 1/2)$ provides better agreement between the piecewise linear constitutive equations and the actual equations of the homogeneous plate.

SOME NUMERICAL RESULTS

To explore the approximation we employ a program which admits piecewise linear paths in strain space (ϵ) and augment the incremental constitutive equations with another equation

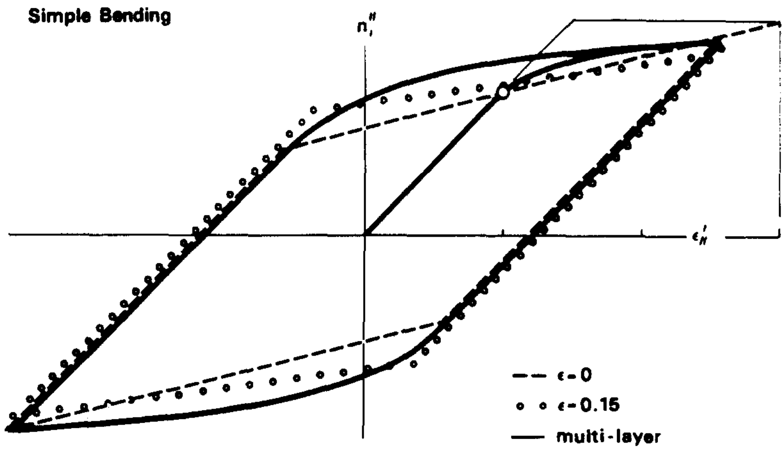

Fig. 2.

which controls the incremental step in stress–strain space; specifically, we define the increment of "arc length"[6]:

$$(\dot{s})^2 = \dot{n}_N{}^{\alpha\beta}\dot{n}^N_{\alpha\beta} + \dot{\epsilon}_N{}^{\alpha\beta}\dot{\epsilon}^N_{\alpha\beta}. \quad \text{(summation on all indices)} \tag{8}$$

To facilitate the computation, eqn (8) is replaced by the approximation which is linear in all *ensuing* increments; incremental arc length and direction ($\boldsymbol{\alpha}$) along the path of strain ($\dot{\boldsymbol{\epsilon}} = \dot{a}\boldsymbol{\alpha}$) is prescribed, and then the increments of stress ($\dot{\mathbf{n}}$) and the amplitude of the strain ($\dot{a}$) are determined by the solution of a linear system in accordance with (1), (3), (6) and (8). During unloading of a layer, the coefficient $B^{\alpha\beta\gamma\eta}$ is suppressed in (3).

Figures 2–5 are plots of bending moment ($n_1{}^{11}$) vs curvature (ϵ^1_{11}), drawn from different histories of strain: simple bending, radial paths of bending and twisting ($\epsilon^1_{11} = \epsilon^1_{12}$), extension and bending ($\epsilon^0_{11} = \epsilon^1_{11}$), and prestrain ($\epsilon^0_{11} = 1$) followed by bending. The solid curves are obtained with the same underlying hypotheses of eqns (1)–(4) and a computational procedure which utilizes 21 stations through the thickness; these smooth curves provide an accurate description under the basic hypotheses. As expected, the discrepancies are greater in the circumstances of dominant bending and twisting, Figs. 2 and 3. Also, the overall fit is better with the small positive value $\epsilon = 0.15$, which introduces a corresponding violation of the initial yield condition. The limiting moment ($n_1{}^{11} = 1/2$) is attained at a finite curvature ($\epsilon^1_{11} \geq 3$), which depends upon the parameter ϵ as shown in Fig. 2.

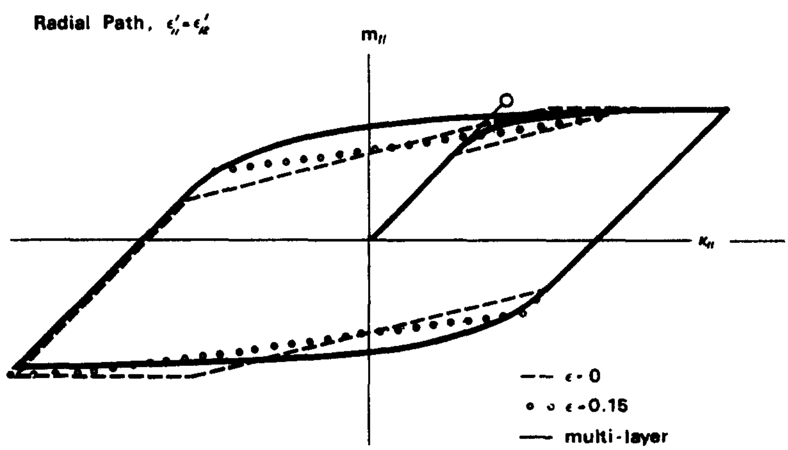

Fig. 3.

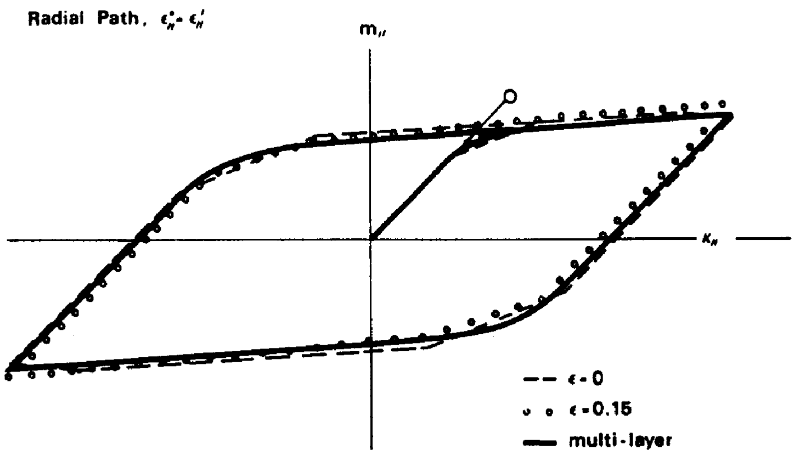

Fig. 4.

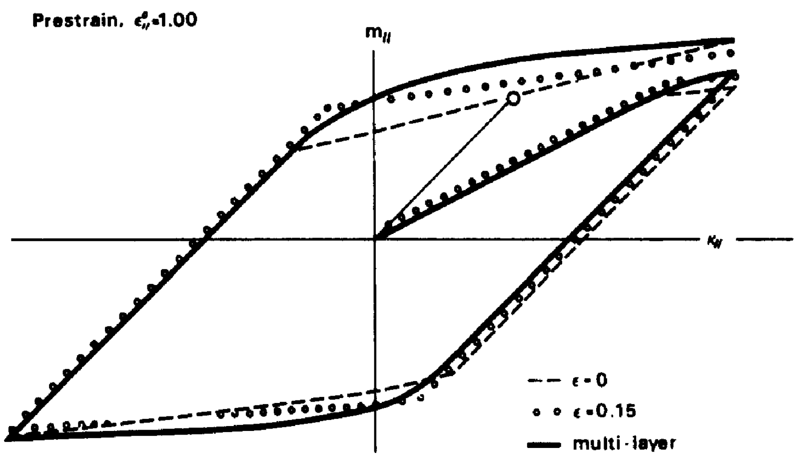

Fig. 5.

SUMMARY

The club sandwich forms the basis of a computational program which provides incremental relations between the 6 components of stress ($n_0{}^{\alpha\beta}, n_1{}^{\alpha\beta}$) and strain ($\epsilon^0_{\alpha\beta}, \epsilon^1_{\alpha\beta}$) in the first approximation. From a practical viewpoint, that program serves as a subroutine in a program for the discrete approximation of plates and shells. The present approximation requires 12 ($\sigma_i{}^{\alpha\beta}$) components of stress at each "node" on the surface and provides piecewise linear approximations as illustrated in Figs. 2–5. Note that the inherent errors of such approximations are reduced if the material exhibits strain hardening.

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